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Functional methods in the generalized Dicke model

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Abstract

The Dicke model describes an ensemble of N identical two-level atoms (qubits) coupled to a single quantized mode of a bosonic field. The fermion Dicke model should be obtained by changing the atomic pseudo-spin operators by a linear combination of Fermi operators. The generalized fermion Dicke model is defined introducing different coupling constants between the single mode of the bosonic field and the reservoir, g_1 and g_2 for rotating and counter-rotating terms, respectively. In the limit $N \to \infty$, the thermodynamic of the fermion Dicke model can be analyzed using the path integral approach with the functional method. The system exhibits a second-order phase transition from normal to superradiance at some critical temperature with the presence of a condensate. We evaluate the critical transition temperature and present the spectrum of the collective bosonic excitations for the general case ($g_1 \neq 0$ and $g_2 \neq 0$). There is a quantum critical behavior when the coupling constants g_1 and g_2 satisfy $g_1 + g_2 = (\omega_0\Omega)^{\frac{1}{2}}$, where ω_0 is the frequency of the mode of the field and Ω is the energy gap between the energy eigenstates of the qubits. Two particular situations are analyzed. First, we present the spectrum of the collective bosonic excitations, in the case $g_1 \neq 0$ and $g_2 = 0$, recovering the well-known results. Second, the case $g_1 = 0$ and $g_2 \neq 0$ is studied. In this last case, it is possible to have a superradiant phase when only virtual processes are introduced in the interaction Hamiltonian. Here also appears a quantum phase transition at the critical coupling $g_2 = (\omega_0\Omega)^{\frac{1}{2}}$, and for larger values for the critical coupling, the system enter in this superradiant phase with a Goldstone mode.

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1. Introduction

In this paper we investigate a generalization of the Dicke model [1], where a single quantized mode of a bosonic field interacts with a reservoir of N identical two-level atoms (qubits) at temperature β^{-1}, using the path integral approach with the functional integration method [2].

We consider the question of how do the counter-rotating terms of the interaction Hamiltonian contribute to the system exhibits a phase transition from normal to superradiance at some critical temperature β_c^{-1}.

In order to apply the path integral approach with the functional method, first it is necessary to change the atomic pseudo-spin operators of the model by a linear combination of Fermi operators to define the generalized fermion Dicke model [3, 4]. Second, the thermodynamic limit ($N \to \infty$) where N is the number of qubits must be taken [5, 6].

We are interested in studying the nonanalytic behavior of thermodynamics quantities near a phase transition in the generalized fermion Dicke model. Introducing different coupling constants, g_1 and g_2, for rotating and counter-rotating terms, respectively, we are able to identify the contribution of each of the processes, real ones and virtual ones in the formation of the condensate. We evaluate the critical transition temperature of the model and present the spectrum of the colective bosonic excitations in the general case, for the case $g_1 \neq 0$ and $g_2 = 0$ and also $g_1 = 0$ and $g_2 \neq 0$. In both cases it appear a critical behavior with Goldstone (gapless) modes. An interesting result is that in the last case, where we only consider the counter-rotating terms in the interaction Hamiltonian, it appears a quantum phase transition [7] at critical coupling $g_2 = (\omega_0 \Omega)^{\frac{1}{2}}$.

At this point we would like to make a summary of results concerning the thermodynamic of the Dicke model. An important result was obtained by Hepp and Lieb [8, 9]. These authors proved that the model presents a second-order phase transition from the normal to the superradiant phase. Also Wang and Hioe [10] obtained some results which agree with those of Hepp and Lieb. The generalized Dicke model, where the counter-rotating terms are also present in the interaction Hamiltonian, was investigated by Hioe [11] and also Duncan [12]. Employing a Holstein–Primakoff mapping [13], Emary and Brandes [14, 15] were able to express the generalized Dicke model in terms of a two-mode bosonic field. These authors discussed the relation between the quantum phase transition and the chaotic behavior that appear in the model for finite N.

We would like to stress that we are not doing any distinction between the elementary unit of quantum information, the qubit and the two-level atom. This paper is organized as follows. In section 2, we discuss the generalized Dicke and fermion model, respectively. In section 3, the path integral with the functional integral method is applied to study the thermodynamic of the generalized fermion Dicke model. Conclusions are given in section 4. In the paper we use $k_B = c = \hbar = 1$.

2. The generalized Dicke model Hamiltonian and the fermion Dicke model

The Hamiltonian of a bosonic quantum system H_S, coupled with the reservoir of qubits, with Hamiltonian H_B, in thermal equilibrium at temperature β^{-1} can be written as

$$H = I_S \otimes H_B + H_S \otimes I_B + H_I, \tag{1}$$

where I_S denotes the identity in the Hilbert space of the quantized bosonic field, I_B denotes the identity in the Hilbert space of the qubit reservoir and H_I is the interaction Hamiltonian.

Using the pseudo-spin operators $\sigma_{(j)}^{+}, \sigma_{(j)}^{-}$ and $\sigma_{(j)}^{z}$ that satisfy the standard angular momentum commutation relations corresponding to spin-$\frac{1}{2}$ operators, the generalized Dicke model is defined by

$$H = I_S \otimes \sum_{j=1}^{N} \frac{\Omega}{2} \sigma_{(j)}^{z} + \omega_0 b^{\dagger} b \otimes I_B + \frac{g_1}{\sqrt{N}} \sum_{j=1}^{N} \left(b\sigma_{(j)}^{+} + b^{\dagger} \sigma_{(j)}^{-} \right) + \frac{g_2}{\sqrt{N}} \sum_{j=1}^{N} \left(b\sigma_{(j)}^{-} + b^{\dagger} \sigma_{(j)}^{+} \right). \tag{2}$$

In the above equation g_1 and g_2 are coupling constants between the qubit and the single mode of the bosonic field. The b and $b^\dagger$ are the boson annihilation and creation operators of mode excitations that satisfy the usual commutation relation rules.

Let us define the fermion Dicke model. Starting from the Hamiltonian of the Dicke model, let us define the Fermi raising and lowering operators $\alpha_i^\dagger, \alpha_i, \beta_i^\dagger$ and β_i that satisfy the anti-commutator relations $\alpha_i\alpha_j^\dagger + \alpha_j^\dagger\alpha_i = \delta_{ij}$ and $\beta_i\beta_j^\dagger + \beta_j^\dagger\beta_i = \delta_{ij}$. We can also define the following bilinear combination of Fermi operators $\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i, \alpha_i^\dagger\beta_i$ and finally $\beta_i^\dagger\alpha_i$. Since the pseudo-spin operators obey the same commutation relations as the above-presented bilinear combination of Fermi operators, we can change the pseudo-spin operators of the Dicke model by the bilinear combination of Fermi operators

$$\sigma_{(i)}^z \longrightarrow \left(\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i\right) \tag{3}$$

$$\sigma_{(i)}^+ \longrightarrow \alpha_i^\dagger\beta_i \tag{4}$$

and finally

$$\sigma_{(i)}^- \longrightarrow \beta_i^\dagger\alpha_i. \tag{5}$$

From now on we use the usual notation instead of the notation stressing the tensor product space of the total Hilbert space of the system. With the substitutions that we defined in equations (3)–(5) we shall call the resulting Hamiltonian as the fermion Dicke model, i.e., H_F. The Hamiltonian of the generalized fermion Dicke model can be written as

$$H_F = \omega_0 b^\dagger b + \frac{\Omega}{2}\sum_{i=1}^{N}\left(\alpha_i^\dagger\alpha_i - \beta_i^\dagger\beta_i\right) + \frac{g_1}{\sqrt{N}}\sum_{i=1}^{N}\left(b\alpha_i^\dagger\beta_i + b^\dagger\beta_i^\dagger\alpha_i\right) + \frac{g_2}{\sqrt{N}}\sum_{i=1}^{N}\left(b^\dagger\alpha_i^\dagger\beta_i + b\beta_i^\dagger\alpha_i\right). \tag{6}$$

In the following section, we consider the problem of defining the partition function of the fermion Dicke model defined by Z_F.

3. The functional integral for the generalized fermion Dicke model

In this section, we consider the problem of defining the partition function Z_F of the generalized fermion Dicke model. First let us define the Euclidean action S of this model, which describes a single quantized mode of the field and the ensemble of N identical qubits. The Euclidean action S is given by

$$S = \int_0^\beta \mathrm{d}\tau \left(b^*(\tau)\frac{\partial}{\partial\tau}b(\tau) + \sum_{i=1}^{N}\left(\alpha_i^*(\tau)\frac{\partial}{\partial\tau}\alpha_i(\tau) + \beta_i^*(\tau)\frac{\partial}{\partial\tau}\beta_i(\tau)\right)\right) - \int_0^\beta \mathrm{d}\tau\, H_F(\tau), \tag{7}$$

where H_F is the full Hamiltonian for the generalized fermion Dicke model given by

$$H_F = \omega_0 b^*(\tau)b(\tau) + \frac{\Omega}{2}\sum_{i=1}^{N}(\alpha_i^*(\tau)\alpha_i(\tau) - \beta_i^*(\tau)\beta_i(\tau)) + \frac{g_1}{\sqrt{N}}\sum_{i=1}^{N}(\alpha_i^*(\tau)\beta_i(\tau)b(\tau) + \alpha_i(\tau)\beta_i^*(\tau)b^*(\tau)) + \frac{g_2}{\sqrt{N}}\sum_{i=1}^{N}(\alpha_i(\tau)\beta_i^*(\tau)b(\tau) + \alpha_i^*(\tau)\beta_i(\tau)b^*(\tau)). \tag{8}$$

Let us define the formal quotient of two functional integrals, i.e., the partition function of the generalized fermion Dicke model and the partition function of the free fermion Dicke model. Therefore, we are interested in calculating the following quantity:

$$\frac{Z_F}{Z_{F_0}} = \frac{\int [\mathrm{d}\eta]\, \mathrm{e}^{S}}{\int [\mathrm{d}\eta]\, \mathrm{e}^{S_0}}, \tag{9}$$

where $S = S(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the Euclidean action of the generalized fermion Dicke model given by equation (7), $S_0 = S_0(b, b^*, \alpha, \alpha^\dagger, \beta, \beta^\dagger)$ is the free Euclidean action for the free single bosonic mode and the free qubits, i.e., the expression of the complete action S taking $g_1 = g_2 = 0$ and finally $[\mathrm{d}\eta]$ is the functional measure. The functional integrals involved in equation (9), are functional integrals with respect to the complex functions $b^*(\tau)$ and $b(\tau)$ and Grassmann Fermi fields $\alpha_i^*(\tau), \alpha_i(\tau), \beta_i^*(\tau)$ and $\beta_i(\tau)$. Since we use thermal equilibrium boundary conditions in the imaginary time formalism, the integration variables in equation (9) obey periodic boundary conditions for the Bose field, i.e., $b(\beta) = b(0)$ and anti-periodic boundary conditions for Fermi fields i.e., $\alpha_i(\beta) = -\alpha_i(0)$ and $\beta_i(\beta) = -\beta_i(0)$.

The free action for the single-mode bosonic field $S_0(b)$ is given by

$$S_0(b) = \int_0^\beta \mathrm{d}\tau \left(b^*(\tau) \frac{\partial b(\tau)}{\partial \tau} - \omega_0 b^*(\tau) b(\tau) \right). \tag{10}$$

Then we can write the action S of the generalized fermion Dicke model, given by equation (7), using the free action for the single-mode bosonic field $S_0(b)$ given by equation (10), plus an additional term that can be expressed in a matrix form. Therefore, the total action S can be written as

$$S = S_0(b) + \int_0^\beta \mathrm{d}\tau \sum_{i=1}^{N} \rho_i^\dagger(\tau) M(b^*, b) \rho_i(\tau), \tag{11}$$

where $\rho_i(\tau)$ is a column matrix given in terms of fermion field operators

$$\rho_i(\tau) = \begin{pmatrix} \beta_i(\tau) \\ \alpha_i(\tau) \end{pmatrix}$$
$$\rho_i^\dagger(\tau) = (\beta_i^*(\tau) \quad \alpha_i^*(\tau)) \tag{12}$$

and the matrix $M(b^*, b)$ is given by

$$M(b^*, b) = \begin{pmatrix} \partial_\tau + \Omega/2 & (N)^{-1/2}(g_1 b^*(\tau) + g_2 b(\tau)) \\ (N)^{-1/2}(g_1 b(\tau) + g_2 b^*(\tau)) & \partial_\tau - \Omega/2 \end{pmatrix}. \tag{13}$$

These fields $b(\tau), \alpha_i(\tau)$ and $\beta_i(\tau)$ can be written as a Fourier expansion. Therefore, we have

$$b(\tau) = \beta^{-1/2} \sum_\omega b(\omega)\, \mathrm{e}^{\mathrm{i}\omega\tau} \tag{14}$$

and

$$\rho_i(\tau) = \beta^{-1/2} \sum_p \rho_i(p)\, \mathrm{e}^{\mathrm{i}p\tau}. \tag{15}$$

Since the field $b(\tau)$ obeys periodic boundary conditions, and the fields $\alpha_i(\tau)$ and $\beta_i(\tau)$ obey anti-periodic boundary conditions, we have that $\omega = \frac{2\pi n}{\beta}$ and $p = \frac{(2n+1)\pi}{\beta}$, where they are the boson and fermion Matsubara frequencies, respectively. Substituting the Fourier expansions into the action given by equation (11) we get

$$S = \sum_\omega (\mathrm{i}\omega - \omega_0) b^*(\omega) b(\omega) + \sum_{p,q} \sum_{i=1}^{N} \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q), \tag{16}$$

where the matrix $M_{pq}(b^*, b)$ is given by

$$M_{pq}(b^*, b) = \begin{pmatrix} (\mathrm{i}p + \Omega/2)\delta_{pq} & (N\beta)^{-1/2}(g_1 b^*(q-p) + g_2 b(p-q)) \\ (N\beta)^{-1/2}(g_1 b(p-q) + g_2 b^*(q-p)) & (\mathrm{i}p - \Omega/2)\delta_{pq} \end{pmatrix}. \tag{17}$$

Using the above results, the ratio between the two functional integrals Z and Z_0, i.e., $\frac{Z}{Z_0}$ is given by

$$\frac{\int [\mathrm{d}\eta(b)] \exp\left(\sum_\omega (\mathrm{i}\omega - \omega_0) b^*(\omega) b(\omega)\right) \int [\mathrm{d}\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q)\right)}{\int [\mathrm{d}\eta(b)] \exp\left(\sum_\omega (\mathrm{i}\omega - \omega_0) b^*(\omega) b(\omega)\right) \int [\mathrm{d}\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(0,0) \rho_i(q)\right)}, \tag{18}$$

where the functional measures $[\mathrm{d}\eta(b)]$ and $[\mathrm{d}\eta(\rho)]$ in the above equation are defined, respectively, by

$$[\mathrm{d}\eta(b)] = \prod_\omega \mathrm{d}b(\omega)\, \mathrm{d}b^*(\omega) \tag{19}$$

and

$$[\mathrm{d}\eta(\rho)] = \prod_{i,p} \mathrm{d}\rho_i(p)\, \mathrm{d}\rho_i^\dagger(p). \tag{20}$$

We need to impose cutoffs over the boson and fermion Matsubara frequencies on these measures. This procedure is necessary to be sure that the ratio between the two functional integrals given by $\frac{Z}{Z_0}$ does not diverge. After all, at the end, we must take these cutoffs to infinity. In order to define the effective action associated with the bosonic mode, we integrate out the fermionic degrees of freedom. The integrals with respect to the Fermi fields are Gaussian and we may integrate over these Grassmann variables. This procedure yields

$$\int [\mathrm{d}\eta(\rho)] \exp\left(\sum_{p,q} \sum_{i=1}^N \rho_i^\dagger(p) M_{pq}(b^*, b) \rho_i(q)\right) = \det{}^N M(b^*, b), \tag{21}$$

where the matrix M is a block matrix of the following form:

$$M(b^*, b) = \begin{pmatrix} \mathrm{i}P + \frac{\Omega}{2} I & (N\beta)^{-1/2} Q^\dagger \\ (N\beta)^{-1/2} Q & \mathrm{i}P - \frac{\Omega}{2} I \end{pmatrix}, \tag{22}$$

where I is the identity matrix and the components of matrix P and Q are

$$\begin{aligned} P_{pq} &= p\delta_{pq} \\ Q_{pq} &= g_1 b(p-q) + g_2 b^*(q-p). \end{aligned} \tag{23}$$

The following change of coordinates can simplify our calculations. Let us change variables in the following way:

$$b(\omega) \quad \to \quad \left(\frac{\pi}{(\omega_0 - \mathrm{i}\omega)}\right)^{1/2} b(\omega) \tag{24}$$

and

$$b^*(\omega) \quad \to \quad \left(\frac{\pi}{(\omega_0 - \mathrm{i}\omega)}\right)^{1/2} b^*(\omega). \tag{25}$$

We must note that equation (25) is not the conjugate of equation (24). Nevertheless, it is not difficult to justify this transformation if we introduce polar coordinates instead of $b(\omega), b^*(\omega)$: $b(\omega) = (\rho(\omega))^{1/2}\, \mathrm{e}^{\mathrm{i}\phi(\omega)}$, $b^*(\omega) = (\rho(\omega))^{1/2}\, \mathrm{e}^{-\mathrm{i}\phi(\omega)}$ and then perform a complex rotation of the integration counter when integrating with respect to $\rho(\omega)$: $\rho(\omega) \to \rho(\omega)[\pi/(\omega_0 - \mathrm{i}\omega)]^{1/2}$. It is easy to see that after these changes of variables the denominator of equation (18), turns out to be equal to unity

$$\int [\mathrm{d}\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega)\right) = 1, \tag{26}$$

so we can express the ratio $\frac{Z}{Z_0}$ by the integral

$$\frac{Z}{Z_0} = \int [\mathrm{d}\eta(b)] \exp(S_{\mathrm{eff}}(b)), \tag{27}$$

where $S_{\mathrm{eff}}(b)$ is the effective action of the bosonic mode which is given by

$$S_{\mathrm{eff}} = -\pi \sum_{\omega} b^*(\omega) b(\omega) + N \ln\det(I + A). \tag{28}$$

The determinant in the above equation is given by

$$\det(I + A) = \det(M^{-1/2}(0,0) M(b^*, b) M^{-1/2}(0,0)) \tag{29}$$

and the matrix A is defined as follows:

$$A = \begin{pmatrix} 0 & B \\ -C & 0 \end{pmatrix}. \tag{30}$$

In the above equation the quantities B and C are matrices with the components given by

$$B_{pq} = \left(\frac{\pi}{\beta N}\right)^{\frac{1}{2}} \left(\mathrm{i}p + \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{g_1 b^*(q-p)}{\sqrt{\omega_0 - \mathrm{i}(q-p)}} + \frac{g_2 b(p-q)}{\sqrt{\omega_0 - \mathrm{i}(p-q)}}\right) \left(\mathrm{i}q - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \tag{31}$$

and

$$C_{pq} = -\left(\frac{\pi}{\beta N}\right)^{\frac{1}{2}} \left(\mathrm{i}p - \frac{\Omega}{2}\right)^{-\frac{1}{2}} \left(\frac{g_1 b(p-q)}{\sqrt{\omega_0 - \mathrm{i}(p-q)}} + \frac{g_2 b^*(q-p)}{\sqrt{\omega_0 - \mathrm{i}(q-p)}}\right) \left(\mathrm{i}q + \frac{\Omega}{2}\right)^{-\frac{1}{2}}. \tag{32}$$

In equation (27) we may go to the limit $\omega_B, \omega_F \to \infty$ and instead of a formal quotient of two infinite functional integrals we shall have only one finite functional integral. This representation turns out to be very useful for obtaining the asymptotic formula for Z/Z_0 at large N. There exists only one stationary phase point at $\beta^{-1} > \beta_c^{-1}$. If $\beta^{-1} < \beta_c^{-1}$, we have a circle of a stationary phase $|b(0)|^2 = \rho_0, b(\omega) = b^*(\omega) = 0$, if $\omega \neq 0$. There also exists an interpolation formula between these asymptotes. The presence of degenerate vacua is a feature of states with spontaneous symmetry breaking. As we will see, gapless excitation will appear.

We shall investigate the integral given by equation (27) for temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$. First let us show that this integral converges. We use the estimate

$$|\det(I + A)| \leqslant \exp\left(\mathrm{Re}(\mathrm{tr}\,A) + \tfrac{1}{2}\,\mathrm{tr}(AA^\dagger)\right). \tag{33}$$

where $\mathrm{Re}(\mathrm{tr}\,A)$ means the real part of $\mathrm{tr}\,A$. The matrix A has the form given by equation (30). Therefore, we find that $\mathrm{tr}\,A = 0$ and $\mathrm{tr}(AA^\dagger) = \mathrm{tr}(BB^\dagger) + \mathrm{tr}(CC^\dagger)$. Therefore, we obtain the estimate

$$\begin{aligned} \frac{Z}{Z_0} &\leqslant \int [\mathrm{d}\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega) b(\omega) + N\,\mathrm{tr}(BB^\dagger) + N\,\mathrm{tr}(CC^\dagger)\right), \\ &\leqslant \int [\mathrm{d}\eta(b)] \exp\left(-\pi \sum_{\omega} b^*(\omega)(1 - a_0(\omega)) b(\omega) \right. \\ &\quad \left. + \pi \sum_{\omega} (b(\omega) c_0(\omega) b(-\omega) + b^*(\omega) c_0(\omega) b^*(-\omega))\right), \end{aligned} \tag{34}$$

where $a_0(\omega)$ and $c_0(\omega)$ are given, respectively, by

$$a_0(\omega) = \frac{g_1^2 + g_2^2}{\beta\left(\omega_0^2 + \omega^2\right)^{1/2}} \sum_{p-q=\omega} \frac{1}{\left(\frac{\Omega^2}{4} + q^2\right)^{1/2}} \frac{1}{\left(\frac{\Omega^2}{4} + p^2\right)^{1/2}} \tag{35}$$

and

$$c_0(\omega) = \frac{\omega_0 g_1 g_2}{\beta\left(\omega_0^2 + \omega^2\right)} \sum_{p-q=\omega} \frac{1}{\left(\frac{\Omega^2}{4} + q^2\right)^{1/2}} \frac{1}{\left(\frac{\Omega^2}{4} + p^2\right)^{1/2}}. \tag{36}$$

Using the measure given in equation (19), we have that $\frac{Z}{Z_0} \leqslant F$, where $F = F_1 F_2$ and F_1 and F_2 are given by

$$F_1 = \int \mathrm{d}b(0)\,\mathrm{d}b^*(0) \exp[-\pi b^*(0)(1 - a_0(0))b(0) + \pi(b(0)c_0(0)b(0) + b^*(0)c_0(0)b^*(0))] \tag{37}$$

and

$$F_2 = \int \prod_{\omega>0} \mathrm{d}b(\omega)\,\mathrm{d}b^*(\omega)\,\mathrm{d}b(-\omega)\,\mathrm{d}b^*(-\omega)$$
$$\times \exp\left[-\pi \sum_{\omega>0} b^*(\omega)(1 - a_0(\omega))b(\omega) - \pi \sum_{\omega>0} b^*(-\omega)(1 - a_0(\omega))b(-\omega)\right.$$
$$\left.\times\ 2\pi \sum_{\omega>0} (b(\omega)c_0(\omega)b(-\omega) + b^*(\omega)c_0(\omega)b^*(-\omega))\right]. \tag{38}$$

Note that in the case of the generalized fermion Dicke model we obtained a Gaussian integral that mixtures positive with negative frequencies. A straightforward calculation gives that the ratio $\frac{Z}{Z_0}$ obeys the following inequality:

$$\frac{Z}{Z_0} \leqslant [(1 - a_0(0) + 2c_0(0))(1 - a_0(0) - 2c_0(0))]^{-1/2}$$
$$\times \prod_{\omega>0} [(1 - a_0(\omega) + 2c_0(\omega))(1 - a_0(\omega) - 2c_0(\omega))]^{-1}. \tag{39}$$

In a similar way to Popov and Fedotov [6] proved, for the case of rotating-wave approximation, we have that $0 < a_0(\omega) + 2c_0(\omega) < a_0(0) + 2c_0(0)$ and $a_0(0) + 2c_0(0) = O(\omega^{-2} \ln \omega)$. Therefore if $a_0(0) + 2c_0(0) < 1$, then equation (39) guarantees convergence of the expression $\frac{Z}{Z_0}$. The condition $a_0(0) + 2c_0(0) = 1$ is the equation for the transition temperature, then we have

$$a_0(0) + 2c_0(0) = \frac{(g_1 + g_2)^2}{\Omega\omega_0} \tanh\left(\frac{\beta_c \Omega}{4}\right) = 1. \tag{40}$$

The inverse of the critical temperature β_c is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\Omega\omega_0}{(g_1 + g_2)^2}\right). \tag{41}$$

Note that there is a quantum phase transition where the coupling constants g_1 and g_2 satisfy $g_1 + g_2 = (\omega_0\Omega)^{\frac{1}{2}}$. For larger values for $(g_1 + g_2)$ the system enters in a superradiant phase. For the case $g_1 = g_2 = \lambda$, Vidal and Dusuel [16] proved that the system undergoes a second-order phase transition at critical coupling $\lambda = \frac{(w_0\Omega)^{\frac{1}{2}}}{2}$; for this purpose they study the behavior for the order parameter of the transition which is the expectation value of the number of excitations associated with the mode of the bosonic field per atom. Emary and Brandes demonstrated that in a general case, $\beta^{-1} \neq 0$, we have also a second-order phase transition [15].

To calculate the asymptotic behavior of the functional integrals at temperatures that satisfy $\beta^{-1} > \beta_c^{-1}$, we can do the following approximation:

$$\det{}^N(I + A) = \det{}^N(I + BC) \to \exp(N\,\mathrm{tr}(BC)). \tag{42}$$

This substitute can be done and we can estimate the error if we divide all the functional space into two domains C_1 and C_2

$$\mathrm{tr}[(BC)(BC)^\dagger] \leqslant (4N)^{-1} \mapsto C_1, \tag{43}$$

$$\mathrm{tr}[(BC)(BC)^\dagger] \geqslant (4N)^{-1} \mapsto C_2. \tag{44}$$

Denoting

$$K_N = \det{}^N(I + A) - \exp(N\,\mathrm{tr}(BC)), \tag{45}$$

for the ratio $\frac{Z}{Z_0}$, we have the following identity:

$$\frac{Z}{Z_0} = \int [\mathrm{d}\eta(b)] \exp\left(-\pi \sum_\omega b^*(\omega)b(\omega) + N\,\mathrm{tr}(BC)\right) + \int_{C_1} [\mathrm{d}\eta(b)] K_N \exp\left(-\pi \sum b^*(\omega)b(\omega)\right) + \int_{C_2} [\mathrm{d}\eta(b)] K_N \exp\left(-\pi \sum b^*(\omega)b(\omega)\right). \tag{46}$$

The first integral of the above equation is Gaussian; let us define it by I_0. We use equations (31) and (32) in order to calculate the trace of BC, i.e., $\mathrm{tr}(BC)$ which is present in the expression I_0. A simple calculation gives

$$I_0 = \int [\mathrm{d}\eta(b)] \exp\left(-\pi \sum_\omega b^*(\omega)(1 - a(\omega))b(\omega) + \pi \sum_\omega (b(\omega)c(\omega)b(-\omega) + b^*(\omega)c(\omega)b^*(-\omega))\right), \tag{47}$$

where $a(\omega)$ and $c(\omega)$ of the above equation are given, respectively, by

$$a(\omega) = \left(\frac{g_1^2(\Omega - \mathrm{i}\omega)^{-1} + g_2^2(\Omega + \mathrm{i}\omega)^{-1}}{(\omega_0 - \mathrm{i}\omega)}\right) \tanh\left(\frac{\beta\Omega}{4}\right) \tag{48}$$

and

$$c(\omega) = \left(\frac{g_1 g_2 \Omega}{(\omega_0^2 + \omega^2)^{1/2}(\Omega^2 + \omega^2)}\right) \tanh\left(\frac{\beta\Omega}{4}\right). \tag{49}$$

Note that to recover the result obtained by Popov and Fedotov [5, 6] we have only to assume $g_2 = 0$. In this case, we have that $c(\omega) = 0$, which simplifies the integration over the mode of the bosonic field in equation (47). Making the integration we obtain that I_0 is given by

$$I_0 = \prod_{\omega} (1 - a(\omega))^{-1}. \tag{50}$$

After this observation let us go back to the general case, where g_1 and g_2 take arbitrary values. The expression I_0 given in equation (47) is a Gaussian integral; this expression is similar to the integral given in equation (34), so following the same steps we get that

$$I_0 = I_0(\omega = 0) \prod_{\omega>0} [c(\omega)^2 - (1 - a(\omega))(1 - a(-\omega))]^{-1}, \tag{51}$$

where $I_0(\omega = 0)$ is the contribution of the condensate given by

$$I_0(\omega = 0) = [(1 - a(0) + 2c(0))(1 - a(0) - 2c(0))]^{-1/2}. \tag{52}$$

It is possible to estimate the error of I_0 which is given by the two last terms of equation (46). For details, see [2]. The errors depend on order N^{-1}. Therefore $\frac{Z}{Z_0}$ can be written as

$$\frac{Z}{Z_0} = [(1 - a(0) + 2c(0))(1 - a(0) - 2c(0))]^{-1/2} \\ \times \prod_{\omega>0} [(1 - a(\omega))(1 - a(-\omega)) - c^2(\omega)]^{-1} + O(N^{-1}). \tag{53}$$

Therefore in the limit $(N \to \infty)$ the equality $\frac{Z}{Z_0} = I_0$ is a good approximation. To find the collective excitation spectrum we have to use the equation

$$c^2(\omega) - (1 - a(\omega))(1 - a(-\omega)) = 0, \tag{54}$$

and making the analytic continuation ($\mathrm{i}\omega \to E$), we obtain the following equation:

$$1 = -\left[\frac{g_1^4 + g_2^4}{(\omega_0^2 - E^2)(\Omega^2 - E^2)}\right] \tanh^2\left(\frac{\beta\Omega}{4}\right) \\ - \left[\frac{g_1^2 g_2^2}{(\omega_0^2 - E^2)}\left(\frac{1}{(\Omega - E)^2} + \frac{1}{(\Omega + E)^2} - \frac{4\Omega^2}{(\Omega^2 - E^2)^2}\right)\right] \tanh^2\left(\frac{\beta\Omega}{4}\right) \\ + \left[\frac{g_1^2(\Omega - E)^{-1} + g_2^2(\Omega + E)^{-1}}{(\omega_0 - E)} + \frac{g_1^2(\Omega + E)^{-1} + g_2^2(\Omega - E)^{-1}}{(\omega_0 + E)}\right] \tanh\left(\frac{\beta\Omega}{4}\right). \tag{55}$$

Solving the above equation for the case $\beta^{-1} = \beta_c^{-1}$, we find the following roots:

$$E_1 = 0 \tag{56}$$

and

$$E_2 = \left(\frac{g_1(\Omega + \omega_0)^2 + g_2(\Omega - \omega_0)^2}{(g_1 + g_2)}\right)^{1/2}. \tag{57}$$

Its low energy state of excitation is a Goldstone mode. Now, let us present the critical temperature and the spectrum of the collective bosonic excitations of the model with the rotating-wave approximation, where $g_1 \neq 0$ and $g_2 = 0$. The result obtained by Popov and Fedotov is recovered, where the equations

$$a(0) = 1 \tag{58}$$

and

$$\frac{g_1^2}{\omega_0 \Omega} \tanh\left(\frac{\beta_c \Omega}{4}\right) = 1 \tag{59}$$

give the inverse of the critical temperature, β_c. It is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\omega_0 \Omega}{g_1^2}\right). \tag{60}$$

In this case also there is a quantum phase transition, i.e., at zero temperature phase transition when $g_1 = (\omega_0 \Omega)^{\frac{1}{2}}$. It is interesting to point out that there are two different ways to analyze the phase transition. The first one is to follow the non-analytic behavior of the thermodynamic quantities as a function of temperature. A different way is to follow the non-analytic behavior of the thermodynamic quantities as a function of the coupling constant strength. Working in this second approach, we may expect that for large coupling constant g_1 there is a superradiant phase. The spectrum of the collective Bose excitations in this case is

$$E_1 = 0 \tag{61}$$

and

$$E_2 = \Omega + \omega_0. \tag{62}$$

Now we will show that it is possible to have a condensate with superradiance in a system of N qubits coupled with one mode of a Bose field where only virtual processes contribute. In the pure counter-rotating wave case, i.e., $g_1 = 0$ and $g_2 \neq 0$, the inverse of the critical temperature, β_c is given by

$$\beta_c = \frac{4}{\Omega} \tanh^{-1}\left(\frac{\omega_0 \Omega}{g_2^2}\right), \tag{63}$$

and the spectrum of the colective Bose excitations is given by

$$E_1 = 0, \tag{64}$$

and

$$E_2 = |\Omega - \omega_0|. \tag{65}$$

A comment is in order concerning the Bose excitations spectrum. In both the cases: using or not the rotating-wave approximation, there is a phase transition. In the case of the rotating-wave approximation $g_1 \neq 0$ and $g_2 = 0$, there is a Goldstone mode ($E = 0$). In the pure counter-rotating wave case $g_1 = 0$ and $g_2 \neq 0$, also there is a Goldstone (gapless) mode. The existence of Goldstone modes and the energy of the other mode was presented for both above-mentioned cases. The spectrum in the general case is given by the Goldstone mode and also by a nonzero energy mode given by equation (57). It is interesting to stress that we obtained a critical behavior in both situations ($g_1 \neq 0$, $g_2 = 0$ and $g_1 = 0$, $g_2 \neq 0$), where the condensate has Goldstone (gapless) modes, with a superradiant state. Therefore, we show that it is possible to have a condensate with superradiance in a system of N qubits coupled with one mode of a Bose field where only virtual processes contribute. It is important to realize that the energy of the non-Goldstone mode in equation (62) is always larger than the energy of the non-Goldstone mode of equation (65), i.e., in the system where the condensate appears due to the virtual processes.

4. Conclusions

In the present paper, we consider the question of how do the counter-rotating terms of the interaction Hamiltonian contribute in the formation of the condensate with a superradiant phase transition in the generalized Dicke model.

An important question is the way of practical realization of the generalized Dicke model in the laboratory. As was stressed by Dimer *et al* [17] it remains as a challenge to provide a physical system where the counter-rotating terms are dominant. Those authors proposed that in cavities with the N qubits, only one mode of quantized field and classical fields (lasers), it is possible to obtain a physical system that corresponds to the generalized Dicke model. Another practical realization of the generalized Dicke model was presented by Wei *et al* [18]. Also, it has been discussed in the literature the possibility of controling the relative importance of the counter-rotating terms in the Jaynes–Cummings model [19], in the laboratory using a ion trap [20]. Another mechanism to explore the importance of virtual processes was proposed by Ford [21] and Ford and Svaiter [22, 23], where the possibility of amplification of the vacuum fluctuations was discussed. These authors studied the renormalized vacuum fluctuations associated with a scalar and electromagnetic field near the focus of a parabolic mirror. Using the geometric optics approximation these authors found that the mirror geometry can produce large vacuum fluctuations near the focus.

An evidence in favor of our results is an experiment where it is possible to control the importance of the counter-rotating terms in the generalized model in such a way that an ideal $g_1 \approx 0$ situation is achieved. Experimental observation of the superradiant phase in this situation will improve our understanding of this phenomenon.

There are some continuations for this paper. The first one is to investigate the model introduced by DiVicenzo [24], at a finite temperature, using also functional integral methods. Also the model which characterizes a intensity-dependent coupling can be discussed using the conventional technique of the path integral and the functional integration method. A similar calculation presented in this paper could be carried out for this model, although the calculations would be somewhat more laborious than the presented in the paper. The generalization of this model with the introduction of the counter-rotating terms deserves further investigations.

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